

# A Natural Basis of States for the Noncommutative Sphere and its Moyal bracket

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## Abstract

An infinite dimensional algebra which is a non-decomposable reducible representation of  $su(2)$  is given. This algebra is defined with respect to two real parameters. If one of these parameters is zero the algebra is the commutative algebra of functions on the sphere, otherwise it is a noncommutative analogue. This is an extension of the algebra normally referred to as the (Berezin) quantum sphere or “fuzzy” sphere. A natural indefinite “inner” product and a basis of the algebra orthogonal with respect to it are given. The basis elements are homogenous polynomials, eigenvectors of a Laplacian, and related to the Hahn polynomials. It is shown that these elements tend to the spherical harmonics for the sphere. A Moyal bracket is constructed and shown to be the standard Moyal bracket for the sphere.

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# I Introduction

The noncommutative or “fuzzy” sphere has been considered by several authors in different contexts. It is an example for a general quantisation procedure [1, 2, 3].

It is also an example often used in noncommutative geometry [4, 5, 6, 7] (see also references within), and in the theory of membranes [8] which has application to supersymmetry. It is studied in relation to coherent states, [9], [10] and as a reduction of the symplectic algebra on  $\mathbb{R}^6$  [11] [12].

Normally the approximation for the algebra of functions on a sphere is in terms of matrices, where the functions on a sphere appear only in the limit as the size of the matrix tends to infinity. In this article, however, we examine a two parameter algebra of polynomials  $\mathcal{P}(\kappa, R)$  with  $\kappa, R \in \mathbb{R}$ . For different values of  $\kappa$  and  $R$  we obtain:

- The commutative algebra of finite sums of harmonics on the sphere (when  $\kappa = 0$ ). In this case  $R$  plays the radius of the sphere.
- The finite matrix representation of  $su(2)$ . When  $\kappa^2(N^2 - 1) = 4R^2$  then  $M_N(\mathbb{C})$  forms a quotient algebra, and  $R^2$  is the Casimir operator.
- A noncommutative algebra of polynomials which is an infinite dimensional representation of  $su(2)$ , for other values of  $\kappa$ .

In section II we introduce the algebra and give a bilinear form on it. In section III we give a basis of  $\mathcal{P}(\kappa, R)$  which is orthogonal with respect to this bilinear form. Some of the basis elements were given previously for the matrix case in [13].

In section IV we give an alternative representation of the elements of  $\mathcal{P}(\kappa, R)$ , and show how the basis elements can be written in terms of Hahn polynomials. We also show that  $\mathcal{P}(\kappa, R)$  may be viewed as an infinite dimensional, reducible, non-decomposable representation of  $su(2)$ , and give some of its ideals.

In section V we look at the commutative case,  $\kappa = 0$ , and show that the basis elements become the standard spherical harmonics. In section VI we calculate the Moyal bracket which is the limit of the commutator as  $\kappa \rightarrow 0$ . We also look at what the limits of the standard operators on  $\mathcal{P}(\kappa, R)$  are.

Finally in the appendix A we draw attention to some facts about the universal enveloping algebra of  $su(2)$  which are needed for some of the proofs.

## Notation and order of proofs

The summation convention is not used in any part of the article. The results in the appendix A are used throughout the article. This section may be read first since no proofs in this section require material from the rest of the article.

## II Definition of the algebra $\mathcal{P}(\kappa, R)$

Given the constants  $\kappa, R \in \mathbb{R}$ , with  $R > 0$ , we define the algebra  $\mathcal{P}(\kappa, R)$  to be

$$\mathcal{P}(\kappa, R) = \{ \text{Free noncommuting algebra of polynomials in } x, y, z \} / \sim \quad (1)$$

where  $\sim$  are the relations:

$$[x, y] \sim i\kappa z, [y, z] \sim i\kappa x, [z, x] \sim i\kappa y, x^2 + y^2 + z^2 \sim R^2 \quad (2)$$

We note that this is a well defined algebra since it is equivalent to the quotient

$$\mathcal{U}(\kappa)/J(R) \quad (3)$$

where  $\mathcal{U} = \mathcal{U}(\kappa)$  is the universal enveloping algebra of the Lie algebra  $su(2)$  and  $J(R)$  is the ideal

$$J(R) = \{(x^2 + y^2 + z^2 - R^2)f \mid f \in \mathcal{U}\} \quad (4)$$

This ideal is two-sided since  $(x^2 + y^2 + z^2 - R^2)$  commutes with all elements in  $\mathcal{U}$ . As shown in section IV we may view  $\mathcal{P}(\kappa, R)$  as a the vector space for a representation of  $su(2)$ . This representation is reducible but not decomposable. (The same of which is true for  $\mathcal{U}(\kappa)$ , the universal enveloping algebra). Any attempt to give  $\mathcal{P}(\kappa, R)$  a Hilbert space structure would make the this representation of  $su(2)$  non-continuous.

Usually  $\kappa$  and  $R$  are implicit and we simply write  $\mathcal{P}$ . We chose the representatives of each equivalent class  $f \in \mathcal{P}$  to be the totally symmetric formally trace-free polynomial in  $\{x, y, z\}$ . This means that we can write  $f \in \mathcal{P}$  as

$$f = \sum_{n=0}^{\text{degree}(f)} \sum_{a_1 \dots a_n=1}^3 f_{a_1 \dots a_n} x^{a_1} x^{a_2} \dots x^{a_n} \quad (5)$$

where  $\{x^1, x^2, x^3\} = \{x, y, z\}$ . Each  $f_{a_1 \dots a_n}$  is completely symmetric in its

indices and satisfies:

$$\sum_{b=1}^3 f_{bba_3a_4\dots a_n} = 0 \quad (6)$$

The condition (6) will be called formally trace-free to distinguish it from the matrix trace. In appendix A we give some more information about the elements of  $\mathcal{U}$  which can be written as totally symmetric polynomials and we define the formal trace of these elements.

There is a natural linear bijection

$$\Psi_{\kappa_1, \kappa_2} : \mathcal{P}(\kappa_1, R) \mapsto \mathcal{P}(\kappa_2, R) \quad (7)$$

given as follows: Let  $f \in \mathcal{P}(\kappa_1, R)$  and  $g \in \mathcal{P}(\kappa_2, R)$  are both written in the totally symmetric formally trace-free form (5). Then  $\Psi_{\kappa_1, \kappa_2}(f) = g$  if and only if  $f_{a_1\dots a_n} = g_{a_1\dots a_n}$  for all indices  $\{a_1, a_2, \dots\}$ . This mapping is principally used when one of the  $\kappa$ 's is zero since it then relates the commutative algebra of functions on the sphere with the noncommutative algebra. It is clear from the definition of  $\Psi_{\kappa_1, \kappa_2}$  that it is not a homomorphism. (i.e. it does not preserve the product on  $\mathcal{P}(\kappa, R)$ ).

Let  $\mathcal{P}^n \subset \mathcal{P}$  be the set of all homogeneous polynomials of order  $n$ , i.e.

$$\mathcal{P}^n = \left\{ \sum_{a_1\dots a_n=1}^3 f_{a_1a_2\dots a_n} x^{a_1} x^{a_2} \dots x^{a_n} \mid f_{a_1a_2\dots a_n} \text{ is totally symmetric and formally trace-free} \right\} \quad (8)$$

Then  $\dim(\mathcal{P}^n) = 2n + 1$ . So as a set

$$\mathcal{P} = \bigoplus_{n=0}^{\infty} \mathcal{P}^n \quad \text{finite sums only} \quad (9)$$

We define the projection

$$\pi_n : \mathcal{P} \mapsto \bigoplus_{r=1}^n \mathcal{P}^r \quad (10)$$

We define the operation of taking the Hermitian conjugate by

$$\dagger : \mathcal{P} \mapsto \mathcal{P}, \quad (ab)^\dagger = b^\dagger a^\dagger, \quad x^\dagger = x, \quad y^\dagger = y, \quad z^\dagger = z, \quad \lambda^\dagger = \bar{\lambda} \quad \text{for } \lambda \in \mathbb{C} \quad (11)$$

There is a sesquilinear form on  $\mathcal{P}$  given by

$$\begin{aligned} \langle \bullet, \bullet \rangle : \mathcal{P} \times \mathcal{P} &\mapsto \mathbb{C} \\ \langle f, g \rangle &= \pi_0(f^\dagger g) \end{aligned} \quad (12)$$

In the section III we give a basis of  $\mathcal{P}^n$  and  $\mathcal{P}$  which are orthogonal with respect to this bilinear form. We also show that this form is Hermitian  $\langle f, g \rangle = \overline{\langle g, f \rangle}$ . However this bilinear form is not positive definite and  $\langle f, f \rangle$

may be positive, negative or zero. It could be called a *degenerate pseudo inner product*.

As stated in the introduction, there are a number of values of  $\kappa$  and  $R$  for which  $\mathcal{P}(\kappa, R)$  is a special algebra. If  $\kappa = 0$  then  $\mathcal{P}(0, R) \cong C_{00}(S^2)$ , the set of finite sums of spherical harmonics. In this case  $\langle \bullet, \bullet \rangle$  does become positive definite and equal to the standard inner product of functions on  $S^2$ . We can then close  $C_{00}(S^2)$  to give  $L^2(S^2)$ . This will be analysed in section V.

If  $\kappa^2(N^2 - 1) = 4R^2$  with  $N \in \mathbb{Z}$ ,  $N \geq 1$  then  $\pi_{N-1}\mathcal{P}(\kappa, R) \cong M_N$  the set of  $N \times N$  matrices. This isomorphism is given explicitly in section III. In this case the bilinear form restricted to  $\mathcal{P}^{N-1}$  is an inner product, while the bilinear form on any other element vanishes.

It is also possible to take the limit  $R \rightarrow \infty$ , with  $\kappa$  constant. This case is dealt with in [4].

### III Orthogonal Basis of $\mathcal{P}(\kappa, R)$

In this chapter we give an explicit orthonormal basis for  $\mathcal{P}^n$  and  $\mathcal{P}$ . As when dealing with representations on  $su(2)$ , it turns out to be very convenient to work with the ladder operators

$$J_+ = x + iy, \quad J_- = x - iy \quad (13)$$

which obey the relations

$$[z, J_+] = \kappa J_+, \quad [z, J_-] = -\kappa J_-, \quad [J_+, J_-] = 2\kappa z, \quad z^2 + \frac{1}{2}J_-J_+ + \frac{1}{2}J_+J_- = \kappa^2 \quad (14)$$

Useful operators on  $\mathcal{P}$  are given by

$$e_z f = [J_z, f] \quad \text{and similarly for } e_x, e_y \quad (15)$$

$$e_{\pm} f = [J_{\pm}, f] \quad (16)$$

$$\Delta = e_z^2 - \kappa e_z + e_+ e_- = e_x^2 + e_y^2 + e_z^2 \quad (17)$$

We will call  $\Delta$  the Laplacian operator.

**Lemma 1** *The Laplacian  $\Delta$  commutes with  $e_+, e_-, e_z$ . With respect to the bilinear form (12),  $e_z$  and  $\Delta$  are self adjoint whilst  $e_-^\dagger = e_+$ .*

**Proof:**

$$\langle e_- f, g \rangle = \pi_0((J_- f)^\dagger g - (f J_-)^\dagger g) = \pi_0(f^\dagger J_+ g - J_+ f^\dagger g)$$

$= \pi_0(f^\dagger J_+ g - f^\dagger g J_+) - \pi_0(e_+(f^\dagger g)) = \langle f, e_+ g \rangle \quad \forall f, g \in \mathcal{P}$

since from corollary 16:  $\pi_0(e_+ f) = 0$  for all  $f \in \mathcal{P}$ . ■

The effects of these operators are calculated below. We will show that the spaces  $\mathcal{P}^n$  are invariant under the operators  $e_x, e_y, e_z, e_\pm, \Delta$ , and that the  $\mathcal{P}^n$  are the orthogonal eigenspaces of  $\Delta$ .

These operators vanish when  $\kappa = 0$  and we obtain the commutative algebra of functions on the sphere. In section VI we give the value  $\kappa^{-1}e_+$  etc and  $\kappa^{-2}\Delta$ . The last of these is the standard Laplacian for functions on the sphere.

In the following theorem we give a basis of  $\mathcal{P}$  which is orthogonal with respect to the bilinear form (12). This is given by  $P_m^n$  where  $n, m \in \mathbb{Z}, n \geq 0, |m| \leq n$ . The  $P_m^n$  are defined to be proportional to  $e_-^{n-m}(J_+^n)$ , and normalised so that  $\langle P_n^m, P_n^m \rangle \in \{1, 0, -1\}$ .

**Theorem 2** *For  $\kappa \neq 0$  there is a basis of  $\mathcal{P}(\kappa, R)$  given by*

$$\{P_n^m(\kappa, R) \mid n, m \in \mathbb{Z}, n \geq 0, |m| \leq n\} \quad (18)$$

where

$$P_n^m(\kappa, R) = \alpha_n \kappa^{m-n} \left( \frac{(n+m)!}{(2n)!(n-m)!} \right)^{1/2} e_-^{n-m}(J_+^n) \quad (19)$$

We shall write  $P_n^m = P_n^m(\kappa, R)$  when there is no doubt about  $\kappa$  and  $R$ . This basis is orthogonal with respect to the bilinear form. The “normalisation” constant  $\alpha_n \in \mathbb{R}$ ,  $\alpha_n > 0$  defined so that  $\langle P_n^m, P_n^m \rangle \in \{1, 0, -1\}$ .

Each  $P_n^m$  can be written as a homogeneous formally trace-free symmetric polynomial in  $(x, y, z)$  of order  $n$ . Thus the set  $\{P_n^m, m = -n \dots n\}$  forms an orthogonal basis for  $\mathcal{P}^n$  and

$$\Psi_{\kappa_1, \kappa_2}(P_n^m(\kappa_1, R)) = P_n^m(\kappa_2, R) \quad (20)$$

Each  $P_n^m$  is an eigenvector of the operators  $e_z$  and  $\Delta$ .

$$e_z P_n^m = \kappa m P_n^m \quad (21)$$

$$\Delta P_n^m = \kappa^2 n(n+1) P_n^m \quad (22)$$

so that the  $\mathcal{P}^n$  are the orthogonal eigenspaces of  $\Delta$ . The ladder operators  $e_+, e_-$  increase or decrease  $m$  so that  $\mathcal{P}^n$  can be viewed as a  $2n+1$  dimensional adjoint representation of  $su(2)$ .

$$e_+ P_n^m = \kappa(n-m)^{1/2}(n+m+1)^{1/2} P_n^{m+1} \quad (23)$$

$$e_- P_n^m = \kappa(n+m)^{1/2}(n-m+1)^{1/2} P_n^{m-1} \quad (24)$$

The effect of taking the Hermitian conjugate is given by

$$(P_n^m)^\dagger = (-1)^m P_n^{-m} \quad (25)$$

**Proof:**

To show  $P_m^n$  is an eigenvector of  $e_z$  (21) we have

$$e_z P_n^m = \alpha_n e_z J_+^m = \alpha_n \sum_{r=0}^{n-1} J_+^r e_z (J_+) J_+^{n-r-1} = \kappa n P_n^m$$

whilst

$$e_z e_- P_n^m = e_- e_z P_n^m + \text{ad}_{[z, J_-]} P_n^m = \kappa m e_- P_n^m - \kappa e_- P_n^m = \kappa(m-1) e_- P_n^m$$

Thus (21) follows by induction.

For the ladder operators  $e_+, e_-$ , (24) is obvious from the definition of  $P_n^m$ . To show (23) we note

$$\begin{aligned} e_+(e_-^{n-m} J_+^n) &= [e_+, e_-^{n-m}] J_+^n = 2\kappa \sum_{r=1}^{n-m} e_-^{r-1} e_z e_-^{n-m-r} J_+^n \\ &= 2\kappa^2 \sum_{r=0}^{n-m-1} e_-^{r-1} e_-^{n-m-r} J_+^n (m+r) = 2\kappa^2 e_-^{n-m-1} J_+^n \sum_{r=0}^{n-m-1} (m+r) \\ &= \kappa^2 (n-m)(n+m+1) e_-^{n-m-1} J_+^n \end{aligned}$$

From the effects of  $e_+, e_-, e_z$  (22) is trivial. Orthogonality is simply an application of  $e_z$ ,  $\Delta$  and lemma 1.

To show that  $\alpha_n$  is independent of  $m$ , we note

$$\begin{aligned} \langle P_n^m, P_n^m \rangle &= \kappa^{-1} (n+m+1)^{-1/2} (n-m)^{-1/2} \langle e_- P_n^{m+1}, P_n^m \rangle \\ &= \kappa^{-1} (n+m+1)^{-1/2} (n-m)^{-1/2} \langle P_n^{m+1}, e_+ P_n^m \rangle \\ &= \langle P_n^{m+1}, P_n^{m+1} \rangle \end{aligned}$$

For the effect of Hermitian conjugation we see that since  $e_z(P_n^0) = 0$  then  $P_n^0$  may be written as a polynomial in  $z$ , and from (14) it is real. This polynomial given explicitly in section IV. Therefore  $(P_n^0)^\dagger = P_n^0$ . Since

$$(e_+ f)^\dagger = [J_+, f]^\dagger = [f^\dagger, J_-] = -e_-(f^\dagger)$$

then

$$(e_+^m f)^\dagger = (-1)^m e_-^m (f^\dagger)$$

By repeated use of (23) and (24) we have

$$P_n^m = \kappa^{-m} \left( \frac{(n+m)!}{(n-m)!} \right)^{1/2} e_+^m (P_n^0)$$

$$P_n^{-m} = \kappa^{-m} \left( \frac{(n+m)!}{(n-m)!} \right)^{1/2} e_-^m(P_n^0)$$

Thus (25) follows.

The proof of the statement that  $\{P_n^m \mid m = -n \dots n\}$  forms a basis for  $\mathcal{P}^n$  is given in appendix A. ■

We note that the elements  $P_n^0$  were discovered by Bayen and Fronsdal [13]. However they don't mention using the ladder operators to get all the elements.

## Finite Representation of $\mathcal{P}$

**Theorem 3** *For the discrete set of  $\kappa$  such that*

$$\kappa^2(N^2 - 1) = 4R^2 \tag{26}$$

*where  $N \in \mathbb{Z}$ ,  $N \geq 1$  there exist a surjective homomorphism*

$$\varphi_N : \mathcal{P}(\kappa, R) \mapsto M_N(\mathbb{C}) \tag{27}$$

*This is the  $N \times N$  representation of  $su(2)$ . It satisfies*

$$\varphi_N(fg) = \varphi_N(f)\varphi_N(g) \quad \forall f, g \in \mathcal{P} \tag{28}$$

$$\varphi_N(f) = 0 \quad \forall f \in \mathcal{P}^n, \quad n \geq N \tag{29}$$

$$\pi_0(f) = \frac{1}{N} \text{tr}(\varphi_N(f)) \quad \forall f \in \mathcal{P} \tag{30}$$

*The bilinear form is a genuine inner product if restricted to  $\pi_{N-1}(\mathcal{P})$ , whilst  $\langle f, g \rangle = 0$  if  $f \in \mathcal{P}^n$ ,  $g \in \mathcal{P}$  and  $n \geq N$ .*

### Proof:

If  $\varphi_N$  is an  $M_N(\mathbb{C})$  representation of  $su(2)$  then  $\varphi_N(x), \varphi_N(y), \varphi_N(z)$  satisfy the same commutation relations, and the Casimir operator is the same as in  $\mathcal{P}(\kappa, R)$ . Thus  $\varphi_N$  is a homomorphism (28).

As in the example below one can write  $\varphi_N(J_+)$  as an upper triangular matrix (with zeros on the diagonal). Therefore  $\varphi_N(J_+^N) = 0$ . Since  $\varphi_N$  is a homomorphism  $\varphi_N(e_-(f)) = 0$  if  $\varphi_N(f) = 0$ , so  $\varphi_N(P_n^m) = 0$  if  $n \geq N$ . Hence (29).

Since can write  $\varphi_N(J_+)$  as an upper triangular matrix  $\text{tr}(\varphi_N(J_+^n)) = 0$  for all  $n > 0$ , and  $\text{tr}(\varphi_N(e_+(f))) = \text{tr}[\varphi_N(J_-), \varphi_N(f)] = 0$  so  $\text{tr}(\varphi_N(P_m^n)) = 0$  for all  $n > 0$ . This gives (30). ■

We now give an explicit representation  $\varphi_N : \mathcal{P} \mapsto M_N(\mathbb{C})$ . This

representation is very useful for calculating formulae. Let  $|N, r\rangle$  with  $r = 0 \dots N - 1$  be the orthogonal basis column vectors which are eigenvectors of  $\varphi_N(z)$ . Since the dimension of the representation is explicit we drop  $\varphi_N$ . Then by rewriting the standard ladder operators we have

$$J_+|N, r\rangle = \kappa(N - r - 1)^{1/2}(r + 1)^{1/2}|N, r + 1\rangle \quad (31)$$

$$J_-|N, r\rangle = \kappa r^{1/2}(N - r)^{1/2}|N, r - 1\rangle \quad (32)$$

$$z|N, r\rangle = \kappa(r - \frac{N-1}{2})|N, r\rangle \quad (33)$$

The dual to  $|N, r\rangle$  is given by  $\langle r, N|$ . The usefulness of these representation is given by the following theorem.

**Theorem 4** *Let  $f$  be a (noncommuting) polynomial in  $\{J_+, J_-, z, \kappa, R\}$ . Using the identities (14) we can put  $f \in \mathcal{P}$ . Thus we can write*

$$f = \sum_{n=0}^{\text{degree}(f)} \sum_{m=-n}^n f_{nm}(\kappa, R) P_n^m \quad (34)$$

where the product  $\alpha_n(\kappa, R)f_{nm}(\kappa, R)$  is a polynomial in  $\kappa$  and  $R$ . The  $f_{nm}$  can be calculated simply from its values when  $\kappa^2(N^2 - 1) = 4R^2$ . That is if  $g_{nm}(\kappa, R)$  is another polynomial and

$$f_{nm}(2R(N^2 - 1)^{-1/2}, R) = g_{nm}(2R(N^2 - 1)^{-1/2}, R) \quad \forall N \in \mathbb{Z}, N \geq 2 \quad (35)$$

Then  $f_{nm}(\kappa, R) = g_{nm}(\kappa, R)$  for all  $\kappa, R$ . This result is independent of the value of  $\alpha_n(\kappa, R)$ .

**Proof:**

The basis term  $P_n^m/\alpha_n$  is a polynomial in  $\{J_+, J_-, z, \kappa, R\}$ . Its value is independent of the actual definition of  $\alpha_n$ . The manipulation of  $f$  into the above form makes sure that  $\alpha_n f_{nm}$  is a polynomial. The second part follows since all polynomials are determined by there value on a finite number of distinct points. ■

We note that given a polynomial  $f(J_+, J_-, z)$  of order  $r$ . We can use the above theorem to calculate  $f_{nm}$ . This process is of the order of  $r^3$ . However if we directly use the equations (14) then the process takes an exponential amount of time. This can be used to get computers to calculate explicit expressions in the  $P_n^m$ .

We can now extend some of the basic facts about the matrix trace for all  $\kappa, R \in \mathbb{R}$  and  $R > 0$

**Corollary 5**

$$\pi_0(fg) = \pi_0(gf) \text{ and } \langle f, g \rangle = \overline{\langle g, f \rangle} \quad \forall f, g \in \mathcal{P}, \quad \kappa, R \in \mathbb{R}, R > 0 \quad (36)$$

**The “normalisation” constant  $\alpha_n$**

**Theorem 6** *Since*

$$\pi_0(J_-^n J_+^n) = \frac{(n!)^2}{(2n+1)!} \prod_{r=1}^n (4R^2 + \kappa^2(1-r^2)) \quad (37)$$

*we define*

$$\sigma_n(\kappa, R) = \text{sign}(\pi_0(J_-^n J_+^n)) \quad (38)$$

$$\alpha_n(\kappa, R) = \begin{cases} |\pi_0(J_-^n J_+^n)|^{-1/2} & \sigma_n(\kappa, R) \neq 0 \\ 1 & \sigma_n(\kappa, R) = 0 \end{cases} \quad (39)$$

*If we let  $N_0$  be the smallest integer greater than or equal to  $(4R^2\kappa^{-2} + 1)^{1/2}$ . i.e.*

$$N_0 = \lceil (4R^2\kappa^{-2} + 1)^{1/2} \rceil \quad (40)$$

*Then the value of  $\sigma_n(\kappa, R)$  is given by*

$$\sigma_n(\kappa, R) = \begin{cases} 1 & n \leq N_0 - 1 \\ (-1)^{n-N_0+1} & n \geq N_0, \text{ and } (4R^2\kappa^{-2} + 1)^{1/2} \notin \mathbb{Z} \\ 0 & n \geq N_0, \text{ and } (4R^2\kappa^{-2} + 1)^{1/2} = N_0 \in \mathbb{Z} \end{cases} \quad (41)$$

*The “normalisation” of  $P_n^m$  is now*

$$\langle P_n^m, P_n^m \rangle = \sigma_n(\kappa, R) \quad (42)$$

**Proof:**

By repeated application of (32) we have

$$\langle r, N | J_-^m J_+^m | N, r \rangle = \kappa^{2m} \frac{(N-r-1)!}{(N-r-m-1)!} \frac{(r+m)!}{r!}$$

thus

$$\begin{aligned} \pi_0(\varphi_N(J_-^m J_+^m)) &= \kappa^{2m} \frac{1}{N} \sum_{r=0}^{N-1} \frac{(N-r-1)!}{(N-r-m-1)!} \frac{(r+m)!}{r!} \\ &= \kappa^{2m} \frac{m!(N-1)!}{N(N-m-1)!} \sum_{r=0}^{N-m-1} \frac{(1+m-N)_r (m+1)_r}{(1-N)_r r!} \\ &= \kappa^{2m} \frac{m!(N-1)!}{N(N-m-1)!} F(1+m-N, m+1; 1-N; 1) \end{aligned}$$

where  $F(a, b; c; z)$  is the hypergeometric function. Since this has only a finite number of terms we may write

$$F(1 + m - N, m + 1; 1 - N; 1) = \lim_{\varepsilon \rightarrow 0} F(1 + m - N, m + 1; 1 - N + \varepsilon; 1)$$

which is also a polynomial. We may then use the standard result

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

This formula is normally only valid when  $\Re(c-a-b) > 0$ , however, here it is valid here since the number of terms is finite. Thus

$$F(1 + m - N, m + 1; 1 - N; 1) = \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(1 - N + \varepsilon)\Gamma(\varepsilon - 2m - 1)}{\Gamma(-m + \varepsilon)\Gamma(-N - m + \varepsilon)} = \frac{m!(N + m)!}{(2m + 1)!(N - 1)!}$$

Which implies

$$\pi_0(J_-^m J_+^m) = \kappa^{2n} \frac{(N + n)!}{N(N - n - 1)!} \frac{(n!)^2}{(2n + 1)!}$$

substituting  $N = (4R^2\kappa^{-2} + 1)^{1/2}$  and using theorem 4 gives (37). The rest of the theorem is derived from this equation. ■

## IV “ $\mathcal{J}$ -expressions”, an Alternative Representations for $\mathcal{P}(\kappa, R)$

Given  $f \in \mathcal{P}$  then we can use the commutation relations (14) to push the  $J_+$  and  $J_-$  to the left of each term. If a  $J_+$  and  $J_-$  appear in one term we can use the Casimir identity to remove both. Thus the resulting terms must either have only  $J_+$ ’s or only  $J_-$ ’s or neither. If we collect all the terms with the same number of  $J_+$  or  $J_-$  as there factors then  $f$  we may written as a sum of terms of the form

$$\{(J_+)^m p(z), p(z), (J_-)^{-m} p(z)\}$$

where  $p(z)$  is a polynomial in  $z$ . By looking at the action of  $e_z$  on each of these possibilities we see that if  $f$  is an eigenvector of  $e_z$  then we may write

$$f = \begin{cases} (J_+)^m p_f^m(z, \kappa, R) & \text{if } e_z f = \kappa m f \text{ and } m > 0 \\ p_f^0(z, \kappa, R) & \text{if } e_z f = 0 \\ (J_-)^{-m} p_f^m(z, \kappa, R) & \text{if } e_z f = \kappa m f \text{ and } m < 0 \end{cases} \quad (43)$$

This will be known as  $\mathcal{J}$  notation. It turns out to be more convenient to consider only the case when  $\mathcal{P}$  has a finite representation, and we therefore write  $p_f^m(z, N)$ . This is a function of  $z$ ,  $N$  and  $\kappa$  with  $\kappa$  being implicit. However

all the expressions can be generalised by substituting  $N = (4R^2\kappa^{-2} + 1)^{1/2}$ , and using theorem 4. This theorem is valid since the definition of  $\kappa^n P_n^m / \alpha_n$  automatically makes it a polynomial in  $(J_\pm, z, R, \kappa)$

**Theorem 7** *Since  $P_n^m$  is a eigenvector of  $e_z$  we have*

$$P_n^m = \begin{cases} (J_+)^m p_n^m(z, N) & \text{for } m > 0 \\ p_n^0(z, N) & \text{for } m = 0 \\ (J_-)^{-m} p_n^m(z, N) & \text{for } m < 0 \end{cases} \quad (44)$$

where for  $m \geq 0$

$$p_n^m(z, N) = \alpha_n (-\kappa)^{n-m} \binom{2n}{n-m}^{-1/2} h_{n-m}^{(m,m)}(z/\kappa + \frac{N-1}{2}, N-m) \quad (45)$$

$$p_n^{-m}(z, N) = \alpha_n (-1)^m (\kappa)^{n-m} \binom{2n}{n-m}^{-1/2} h_{n-m}^{(m,m)}(z/\kappa - m + \frac{N-1}{2}, N-m) \quad (46)$$

where  $h_n^{(\alpha,\beta)}(x, N)$  follows the notation of [14, chapter 2]. These are the Hahn Polynomials. This has an explicit formulation in terms of generalised hypergeometric functions:

$$p_n^m(z, N) = \alpha_n (-\kappa)^{n-m} \binom{2n}{n-m}^{-1/2} \frac{(m+1)_{n-m} (m+1-N)_{n-m}}{(n-m)!} \times {}_3F_2(m-n, -z/\kappa - \frac{N-1}{2}, n+m+1; m+1, m+1-N; 1) \quad (47)$$

where  $m \geq 0$ .

Before proving this, let's start with a little lemma.

**Lemma 8** *For a polynomial  $p(z)$  and  $m \in \mathbb{Z}^+$  a positive integer*

$$p(z) J_+^m = J_+^m p(z + m\kappa) \quad (48)$$

$$p(z) J_-^m = J_-^m p(z - m\kappa) \quad (49)$$

$$J_+^m J_-^m = \prod_{s=0}^{m-1} \left( R^2 - (z - s\kappa)(z - (s+1)\kappa) \right) \quad (50)$$

$$J_-^m J_+^m = \prod_{s=0}^{m-1} \left( R^2 - (z + s\kappa)(z + (s+1)\kappa) \right) \quad (51)$$

**Proof:**

From (14) we have  $zJ_+ = J_+(z + \kappa)$ . Thus  $zJ_+^m = J_+^m(z + m\kappa)$  and  $z^p J_+^m = J_+^m(z + m\kappa)^p$ , hence result.

For the other equations we note

$$J_+^m J_-^m = J_+^{m-1} (R^2 - z(z - \kappa)) J_-^{m-1} = J_+^{m-1} J_-^{m-1} (R^2 - (z - (m-1)\kappa)(z - m\kappa))$$

■

**Proof of theorem 7:**

Given two basis harmonics  $P_{n_1}^{m_1}, P_{n_2}^{m_2} \in \mathcal{P}$  then we have  $\langle P_{n_1}^{m_1}, P_{n_2}^{m_2} \rangle = 0$  if  $m_1 \neq m_2$  or  $n_1 \neq n_2$ . Writing these in  $\mathcal{J}$  form we have for  $m_1 \neq m_2$  or  $n_1 \neq n_2$  and  $m_1, m_2 \geq 0$

$$\pi_0(p_{n_1}^{m_1}(z, N) J_-^{m_1} J_+^{m_2} p_{n_2}^{m_2}(z, N)) = 0$$

thus

$$\sum_{r=0}^{N-1} \langle r, N | p_{n_1}^{m_1}(z, N) J_-^{m_1} J_+^{m_2} p_{n_2}^{m_2}(z, N) | N, r \rangle = 0$$

It is clear this is satisfied for  $m_1 \neq m_2$ . It is also obvious that the summand vanished if  $r + m > N - 1$  Taking  $m_1 = m_2 = m \geq 0$  and  $n_1 \neq n_2$  we have

$$\sum_{r=0}^{N-m-1} p_{n_1}^m(\kappa(r - \frac{N-1}{2}), N) p_{n_2}^m(\kappa(r - \frac{N-1}{2}), N) \langle r, N | J_-^m J_+^m | N, r \rangle$$

Now

$$\langle r, N | J_-^m J_+^m | N, r \rangle = \kappa^{2m} \left( \frac{(N-r-1)!}{(N-r-m-1)!} \frac{(r+m)!}{r!} \right)$$

is precisely the weight function for the Hahn Polynomials  $h_{n'}^{(\alpha, \beta)}(r, N')$  where  $\alpha = m, \beta = m, n' = n - m$  and  $N' = N - m$ . Now to get the correspondence between the functions we look at coefficient of the highest order in  $r$ . It is easy to show

$$e_-(J_+^m z^p) = -\kappa(p+2m) J_+^{m-1} (z^{p+1} + O(z, p))$$

where  $O(z, p)$  is a polynomial in  $z$  of order  $p$  or less. So

$$e_-^p(J_+^m) = (-\kappa)^p \frac{(2n)!}{(2n-p)!} J_+^{n-p} (z^p + O(z, p-1))$$

so

$$P_n^m = \alpha_n (-1)^{n-m} \binom{2n}{n-m}^{1/2} J_+^m (z^{n-m} + O(z, n-m-1))$$

so  $p_n^m(z, N)$  is a Polynomial in  $z$  of order  $n - m$ . This is the same order  $h_{n-m}^{(m,m)}(r, N)$ . From [14, page 42] we have

$$h_{n-m}^{(m,m)}(z/\kappa + \frac{N-1}{2}, N) = \binom{2n}{n-m} (z^{n-m} + O(z, n-m-1))$$

hence (45). Expression (47) follows from the literature.

Finally (46) is simply an application of lemma 8. ■

## The reducibility and ideals of $\mathcal{P}(\kappa, R)$

**Lemma 9** *For any  $f \in \mathcal{P}$  we define*

$$\begin{aligned} \omega_n : \mathcal{P} &\mapsto \mathcal{P} \\ \omega_n(f) &= \sum_{m=-n}^n (P_n^m)^\dagger f P_n^m \end{aligned} \quad (52)$$

*then  $\omega_n$  is a self-adjoint operator on  $\mathcal{P}$  with respect to the bilinear form. It commutes with the operators  $e_+, e_-, e_z, \Delta$ . It is diagonal with respect to the basis elements  $P_n^m$ , and the eigenvalues depend only on  $n$  so we can write*

$$\omega_n(f) = \omega_{na} f \quad \forall f \in \mathcal{P}^a \quad (53)$$

*where  $\omega_{na}$  is a real polynomial of  $(n, \kappa)$ .*

$$\sum_{m=-n}^n (P_n^m)^\dagger P_n^m = \omega_n(1) = \omega_{n0} = \sigma_n(\kappa, R)(2n+1) \quad (54)$$

### Proof:

Self-adjointness follows from the definition of  $\omega_n$  and corollary 5. Whilst the fact that it commutes with  $e_\pm, e_z, \Delta$  follows from direct substitution. Since  $\omega_p(P_n^m)$  is a polynomial then by the operations of  $e_z$  and  $\Delta$  it is clear that it must be proportional to  $P_n^m$ . Furthermore since  $\omega_p$  commutes with  $e_-$ , then the eigenvector for the space spanned by  $P_n^m$  must be independent of  $m$ . Hence (53). Since  $\omega_p(1) \in \mathcal{P}^0$  then (54) follows by considering  $\pi_0(\omega_p(1))$ . ■

**Lemma 10** *For all  $\kappa, R \in \mathbb{R}$ ,  $\mathcal{P}(\kappa, R)$  has at least one proper left ideal given by  $I = \{fz \mid f \in \mathcal{P}\}$ . Also  $\mathcal{P}(\kappa, R)$  has a proper two sided ideal if and only if  $(4R^2\kappa^{-2} + 1)^{1/2} \in \mathbb{Z}$*

**Proof:**

To see that  $I$  is a proper left ideal we note that  $1 \notin I$ . For assume there exists  $f \in \mathcal{P}$  such that  $fz = 1$ , then writing  $f$  in  $\mathcal{J}$ -notation we have

$$fz = \sum_{a=0}^{\text{degree}(f)} J_+^a p_a(z)z + \sum_{a=1}^{\text{degree}(f)} J_-^a p_{-a}(z)z = 1$$

where  $p_a(z)$  is a polynomial in  $z$ . By looking at the operation of  $e_z$  implies  $p_a(z) = 0$  for  $a \neq 0$ , whilst  $p_0(z)z = 1$  which is impossible. Thus  $I$  is a left ideal of  $\mathcal{P}$ .

We note however that if  $\mathcal{P}$  contained infinite (unbounded) sums then there is a solution  $f = \sum_{n=0}^{\infty} f_n P_n^0$  such that  $fz = 1$ . This expression cannot be written in  $\mathcal{J}$ -notation.

If  $4R^2 = \kappa^2(N^2 - 1)$ , for some  $N \in \mathbb{Z}$ , then from theorem 3 the subspace  $\oplus_{r=N}^{\infty} \mathcal{P}^r \subset \mathcal{P}(\kappa, R)$  is a two sided ideal. This is because it is the kernel of  $\varphi_N$ . Otherwise let  $I \subset \mathcal{P}$  be a two sided ideal and  $f \in I$ . Then by the operation of  $e_z$  and  $\Delta$  we can show there is a basis element  $P_n^m \in I$ . By application of  $e_{\pm}$  we have  $\mathcal{P}^n \subset I$  for some  $n$ . From (54) we have  $\sigma_n(\kappa, R)(2n+1) \in I$ . Since  $(4R^2\kappa^{-2} + 1)^{1/2} \notin \mathbb{Z}$  we have from theorem 6,  $\sigma_n(\kappa, R) \neq 0$  so  $1 \in I$ . ■

**Theorem 11** *The map given by*

$$\begin{aligned} \rho : su(2) &\mapsto \{f : \mathcal{P} \mapsto \mathcal{P} \mid f \text{ is linear}\} \\ \rho(a)(f) &= af \quad a \in su(2), f \in \mathcal{P} \end{aligned} \tag{55}$$

*may be viewed as an infinite dimensional representation of  $su(2)$ . This representation is always reducible but not decomposable. It is “Hermitian” in that it respects the Hermitian conjugate with defined by the bilinear form*

$$\langle \rho(a)f, g \rangle = \langle f, \rho(a^\dagger)g \rangle \quad \forall a \in su(2), f, g \in \mathcal{P} \tag{56}$$

**Proof:**

The subspace  $I \in \mathcal{P}$  given in lemma 10 is invariant under the action of  $\rho$ . However  $\mathcal{P}$  is not decomposable because from the action of  $\rho$  on the element  $1 \in \mathcal{P}$  one can generate  $\mathcal{P}$ . The Hermitian conjugate is by direct substitution. ■

We note that the universal enveloping,  $\mathcal{U}(\kappa)$  is also reducible but not decomposable. Any attempt to give  $\mathcal{P}(\kappa, R)$  a Hilbert space structure would

mean the action of  $su(2)$  were non-continuous operators.

## V The Commutative Case $\kappa = 0$

As mentioned in the introduction the algebras  $\mathcal{P}(\kappa, R)$  is a commutative algebra when  $\kappa = 0$  and is isomorphic to the algebra of functions on the sphere. In this chapter we will show that  $\mathcal{P}(0, R) = C_{00}(S^2)$  the set of finite sums of spherical harmonics,

To make this isomorphism explicit we write

$$\begin{aligned} x|_{\kappa=0} &= \mathbf{x} = R \sin \phi \sin \theta \\ y|_{\kappa=0} &= \mathbf{y} = R \cos \phi \sin \theta \\ z|_{\kappa=0} &= \mathbf{z} = R \cos \theta \end{aligned} \quad (57)$$

To distinguish elements of  $\mathcal{P}(\kappa, R)$  with  $\kappa \neq 0$  from the elements of  $C_{00}(S^2)$  the latter are written in bold when there may be doubt. From (13) we have

$$\begin{aligned} \mathbf{J}_+ &= ie^{-i\phi} R \sin \theta \\ \mathbf{J}_- &= -ie^{i\phi} R \sin \theta \end{aligned} \quad (58)$$

From (39) we have

$$\alpha_n|_{\kappa=0} = \frac{((2n+1)!)^{1/2}}{n!} (2R)^{-n} \quad (59)$$

From (26) we may think of the case  $\kappa = 0$  as the limit as  $N \rightarrow \infty$ . Near this limit (i.e. for large  $N$ )

$$\kappa \sim 2R/N \quad (60)$$

The definition for  $P_n^m$  (19) is not valid in the case  $\kappa = 0$ . We therefore define them as the limit

$$P_n^m(0, R) = \lim_{\kappa \rightarrow 0} \Psi_{\kappa,0}(P_n^m(\kappa, R)) \quad (61)$$

**Theorem 12** *In the case  $\kappa = 0$  the “fuzzy” spherical harmonics become the standard spherical harmonics*

$$P_n^m|_{\kappa=0} = (-1)^n \left( \frac{(n+m)!(2n+1)}{(n-m)!} \right)^{1/2} e^{-im\phi} P_n^{-m}(\cos \theta) = (-1)^n Y_n^{-m}(\theta, \phi) \quad (62)$$

where  $P_n^m(\mathbf{z}/R)$  are the Associated Legendre functions, and  $Y_n^{-m}(\theta, \phi)$  are the orthonormal harmonics on the sphere. So  $\mathcal{P}(0, R) = C_{00}(S^2)$ , the set of finite sums of spherical harmonics. The bilinear form on  $\mathcal{P}$  becomes the standard

inner product on  $C_{00}(S^2)$ :

$$\langle f, g \rangle \rightarrow \langle \mathbf{f}, \mathbf{g} \rangle_{S^2} = \frac{1}{4\pi R^2} \int_{S^2} \bar{\mathbf{f}} \mathbf{g} \sin \theta d\phi d\theta \quad (63)$$

**Proof:**

From (45), (61) and theorem 4, we have for  $m \geq 0$

$$\begin{aligned} P_n^m|_{\kappa=0} &= \lim_{N \rightarrow \infty} (P_n^m) \\ &= \lim_{N \rightarrow \infty} \left( J_+^m \alpha_n (-\kappa)^{n-m} \binom{2n}{n-m}^{-1/2} h_{n-m}^{(m,m)}(z/\kappa + \frac{N-1}{2}, N-m) \right) \\ &= (iR \sin \theta)^m e^{-im\phi} \frac{((2n+1)!)^{1/2}}{n!} (2R)^{-n} (-2R)^{n-m} \binom{2n}{n-m}^{-1/2} \times \\ &\quad \lim_{N \rightarrow \infty} \left( N^{m-n} h_{n-m}^{(m,m)}(z/\kappa + \frac{N-1}{2}, N-m) \right) \end{aligned}$$

From [14, page 46] this is given by

$$\lim_{N \rightarrow \infty} (P_n^m) = (iR \sin \theta)^m e^{-im\phi} \frac{((2n+1)!)^{1/2}}{n!} (2R)^{-n} (-2R)^{n-m} \binom{2n}{n-m}^{-1/2} P_{n-m}^{(m,m)}(\cos \theta)$$

where  $P_{n-m}^{(m,m)}(z/R)$  is the Jacobi Polynomial. This is related to the Associated Legendre functions by

$$P_{n-m}^{(m,m)}(\cos \theta) = \frac{n!}{(n-m)!} (2i \sin \theta)^{-m} P_n^{-m}(\cos \theta)$$

Hence (62). For  $m > 0$  we note that taking the limit of (25)

$$P_n^{-m}|_{\kappa=0} = (-1)^m \overline{P_n^m}|_{\kappa=0} = (-1)^n Y_n^m(\theta, \phi)$$

It is clear now that  $\mathcal{P}(0, R) = C_{00}$ .

If  $f = \sum_{nm} f_{nm} Y_n^m \in \mathcal{P}(0, R)$  then  $\pi_0(f) = f_{00}$ . However

$$\frac{1}{4\pi R^2} \int Y_n^m(\theta, \phi) \sin \theta d\phi d\theta = \begin{cases} 1 & m = 0 \text{ and } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

so (63). ■

## VI The Moyal Bracket

The Moyal Bracket is defined by the limit of the commutator of two elements

$$\{\bullet, \bullet\} : C_{00}(S^2) \times C_{00}(S^2) \mapsto C_{00}(S^2)$$

$$\{\mathbf{f}, \mathbf{g}\} = \lim_{\kappa \rightarrow 0} \left( \frac{1}{i\kappa} [\Psi_{0,\kappa}(\mathbf{f}), \Psi_{0,\kappa}(\mathbf{g})] \right) \quad (64)$$

**Theorem 13** *If  $\mathbf{f}, \mathbf{g} \in C_{00}(S^2)$  then we have the the Moyal bracket is the natural bracket arising from the symplectic form on  $S^2$ .*

$$\{\mathbf{f}, \mathbf{g}\} = \frac{1}{R \sin \theta} \left( \frac{\partial \mathbf{f}}{\partial \phi} \frac{\partial \mathbf{g}}{\partial \theta} - \frac{\partial \mathbf{f}}{\partial \theta} \frac{\partial \mathbf{g}}{\partial \phi} \right) \quad (65)$$

**Proof:**

Since we are dealing with only finite sums of basis elements we need not worry about limits. Since the Moyal bracket is linear in both terms we need only consider its effect on basis elements. Let  $f = \Psi_{0,\kappa}(\mathbf{f})$  and  $g = \Psi_{0,\kappa}(\mathbf{g})$  be eigenvectors of  $e_z$ . For this proof we write  $\partial_\phi = \partial/\partial\phi$ . We note that for a polynomial  $p(\mathbf{z})$

$$\begin{aligned} \partial_\phi \left( \mathbf{J}_+^m p(\mathbf{z}) \right) &= -im \mathbf{J}_-^m p(\mathbf{z}) \\ \partial_\phi \left( \mathbf{J}_-^m p(\mathbf{z}) \right) &= im \mathbf{J}_+^m p(\mathbf{z}) \\ \frac{1}{R \sin \theta} \partial_\theta \left( \mathbf{J}_+^m p(\mathbf{z}) \right) &= -\mathbf{J}_+^m \left( p'(\mathbf{z}) - \frac{m\mathbf{z}}{R^2 - \mathbf{z}^2} p(\mathbf{z}) \right) \\ \frac{1}{R \sin \theta} \partial_\theta \left( \mathbf{J}_-^m p(\mathbf{z}) \right) &= -\mathbf{J}_-^m \left( p'(\mathbf{z}) - \frac{m\mathbf{z}}{R^2 - \mathbf{z}^2} p(\mathbf{z}) \right) \end{aligned}$$

It is necessary to consider separately the cases that the eigenvalues of  $f$  and  $g$  have (1) the same sign and (2) different signs. Let us first consider the case when the eigenvalues of  $f$  and  $g$  have positive sign. Then we can write

$$f = J_+^a p(\mathbf{z}) \text{ , and } g = J_+^b q(\mathbf{z})$$

where  $a, b \geq 0$  and  $p(\mathbf{z}), q(\mathbf{z})$  are polynomials. Then

$$\begin{aligned} [f, g] &= J_+^a p(\mathbf{z}) J_+^b q(\mathbf{z}) - J_+^b q(\mathbf{z}) J_+^a p(\mathbf{z}) \\ &= J_+^{a+b} (p(\mathbf{z} + \kappa b) q(\mathbf{z}) - p(\mathbf{z}) q(\mathbf{z} + \kappa a)) \\ &= J_+^{a+b} \left( (p(\mathbf{z} + \kappa b) - p(\mathbf{z})) q(\mathbf{z}) - p(\mathbf{z}) (q(\mathbf{z} + \kappa a) - q(\mathbf{z})) \right) \end{aligned}$$

In the limit as  $\kappa \rightarrow 0$  we have  $J_+ \rightarrow \mathbf{J}_+$ ,  $p(\mathbf{z}) \rightarrow \mathbf{p}(\mathbf{z})$  and  $1/\kappa(p(\mathbf{z} + b\kappa) - p(\mathbf{z})) \rightarrow b\mathbf{p}'(\mathbf{z})$ . So

$$\begin{aligned} \lim_{\kappa \rightarrow 0} \left( \frac{1}{i\kappa} [f, g] \right) &= -i \mathbf{J}_+^{a+b} (b\mathbf{p}'(\mathbf{z}) \mathbf{q}(\mathbf{z}) - a\mathbf{p}(\mathbf{z}) \mathbf{q}'(\mathbf{z})) \\ &= (R \sin \theta)^{-1} (\partial_\phi \mathbf{f} \partial_\theta \mathbf{g} - \partial_\theta \mathbf{f} \partial_\phi \mathbf{g}) \end{aligned}$$

Hence true in this case. If the eigenvalues of  $f$  and  $g$  both have negative sign then we note that

$$f^\dagger \rightarrow \overline{f}$$

and

$$[f, g]/(i\kappa) = [g^\dagger, f^\dagger]^\dagger/(i\kappa) \rightarrow \overline{\{\overline{f}, \overline{g}\}} = \{\overline{f}, \overline{g}\}$$

Now consider case (2) we write

$$\begin{aligned} f &= J_+^a p(z), \quad g = J_-^b q(z), \quad \text{and } \varrho_-^b(z) = J_+^b J_-^b, \quad \varrho_+^b(z) = J_-^b J_+^b \\ \text{where } a, b &\geq 0 \text{ and } p(z), q(z) \text{ are polynomials. Consider first } a \geq b \\ [f, g] &= J_+^a p(z) J_-^b q(z) - J_-^b q(z) J_+^a p(z) \\ &= J_+^a J_-^b p(z - \kappa b) q(z) - J_-^b J_+^a p(z) q(z + \kappa a) \\ &= J_+^{a-b} \varrho_-^b(z) p(z - \kappa b) q(z) - J_+^{a-b} \varrho_+^b(z + (a - b)\kappa) p(z) q(z + \kappa a) \\ &= J_+^{a-b} \left( \varrho_-^b(z) (p(z - \kappa b) - p(z)) q(z) + \varrho_-^b(z) p(z) (q(z) - q(z + \kappa a)) \right. \\ &\quad \left. + (\varrho_-^b(z) - \varrho_+^b(z)) p(z) q(z + \kappa a) + (\varrho_+^b(z) - \varrho_+^b(z + \kappa(a - b))) p(z) q(z + \kappa a) \right) \end{aligned}$$

In the limit  $\kappa \rightarrow 0$  this becomes

$$\begin{aligned} \lim_{\kappa \rightarrow 0} \left( \frac{1}{i\kappa} [f, g] \right) &= \mathbf{J}_+^{a-b} \left( (R^2 - \mathbf{z}^2)^b \mathbf{p}'(\mathbf{z}) (-b) \mathbf{q}(\mathbf{z}) + (R^2 - \mathbf{z}^2)^b \mathbf{p}(\mathbf{z}) (-a) \mathbf{q}'(\mathbf{z}) \right. \\ &\quad \left. + 2(R^2 - \mathbf{z}^2)^{b-1} b^2 \mathbf{z} \mathbf{p}(\mathbf{z}) \mathbf{q}(\mathbf{z}) - (a - b) \mathbf{p}(\mathbf{z}) \mathbf{q}(\mathbf{z}) \frac{d}{dz} (R^2 - \mathbf{z}^2)^b \right) \\ &= \mathbf{J}_+^{a-b} \left( (R^2 - \mathbf{z}^2)^b (-b \mathbf{p}'(\mathbf{z}) \mathbf{q}(\mathbf{z}) - a \mathbf{p}(\mathbf{z}) \mathbf{q}'(\mathbf{z})) + (R^2 - \mathbf{z}^2)^{b-1} 2zab \mathbf{p}(\mathbf{z}) \mathbf{q}(\mathbf{z}) \right) \\ &= (R \sin \theta)^{-1} \left( \partial_\phi (\mathbf{J}_+^a \mathbf{p}(\mathbf{z})) \partial_\theta (\mathbf{J}_-^b \mathbf{q}(\mathbf{z})) - \partial_\phi (\mathbf{J}_-^b \mathbf{q}(\mathbf{z})) \partial_\theta (\mathbf{J}_+^a \mathbf{p}(\mathbf{z})) \right) \end{aligned}$$

Likewise if  $b > a$  then we consider  $[f^\dagger, g^\dagger]$  as before. ■

We now wish to consider how we can extend this theorem to cover the largest possible subset of  $L^2(S^2)$ . For our case a sufficient extension to  $C_{00}(S^2)$  is given by the set

$$\left\{ f = \sum_{nm} f_{nm} Y_n^m \in L^2(S^2) \mid |f_{nm}| \sim n^{-3} \right\} \quad (66)$$

This is because in this case  $\partial_\phi f \in L^2(S^2)$  and  $(\sin \theta)^{-1} \partial_\theta f \in L^2(S^2)$ . Hence the right hand side of (65) is defined.

## Limit of the operators as $\kappa \rightarrow 0$

As already mentioned the operations  $e_z, e_\pm, \Delta$  from (15) to (17) mean they identically vanish if  $\kappa = 0$ . We therefore calculate the first non-vanishing term in there expansions. We see that  $e_x, e_y, e_z, e_+, e_-$  are vector fields whilst  $\Delta$  is a second-order differential operator corresponding to Laplacian.

**Theorem 14** *In the limit  $\kappa \rightarrow 0$  we have*

$$(i\kappa)^{-1}e_x \rightarrow \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi \quad (67)$$

$$(i\kappa)^{-1}e_y \rightarrow -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi \quad (68)$$

$$(i\kappa)^{-1}e_z \rightarrow \partial_\phi \quad (69)$$

$$(i\kappa)^{-1}e_+ \rightarrow e^{-i\phi} (\partial_\theta - i \cot \theta \partial_\phi) \quad (70)$$

$$(i\kappa)^{-1}e_- \rightarrow e^{i\phi} (\partial_\theta + i \cot \theta \partial_\phi) \quad (71)$$

and as one would expect

$$(i\kappa)^{-1}e_x(\mathbf{y}) \rightarrow \mathbf{z} \text{ and cyclic permutations} \quad (72)$$

These are not independent since

$$\mathbf{x}e_x + \mathbf{y}e_y + \mathbf{z}e_z = 0 \quad (73)$$

Also the “fuzzy” Laplace operator tends to the usual Laplace operator on the sphere.

$$-\kappa^{-2}\Delta \rightarrow \partial_\theta^2 + \cot \theta \partial_\theta + (\sin \theta)^{-2} \partial_\phi^2 \quad (74)$$

**Proof:**

The expressions for  $e_+, e_-, e_z$  come by substituting  $\mathbf{J}_+, \mathbf{J}_-, \mathbf{z}$  as one of the term in the Moyal bracket. The other expressions are derived from these. ■

## VII Discussion

This work forms a basis for the investigation into the differential and connection structures on the fuzzy sphere. (Follow references in [5]). Since  $\mathcal{P}^n$  is a  $2n + 1$  dimensional representation of  $su(2)$  we may consider these representing Bosonic states. One should be able to create another basis of the Fermionic states. This may look like  $P_n^m$  with  $m$  and  $n$  positive odd multiples of a half.

It would be useful to know how these results can be extended for other algebras. For  $su(3)$  one would consider replacing  $J_{\pm}$  with  $u_{\pm}$  and  $v_{\pm}$ , where  $u_{-}, v_{-}$  are the root system. In this case we would have a basis something like

$$e_{u_{-}}^a e_{v_{-}}^b (u_{+}^c v_{+}^d)$$

with some relation for the  $a, b, c, d$ . The case of  $su(n)$  would be equivalent using the root system. We might be able to extend this to all Lie algebras of compact Lie group.

This article has demonstrated how one can use a quotient of a free noncommuting algebra on a finite set of elements to examine a geometry. For existence of an exterior algebra this quotient algebra must form a “generalised algebra” [15]. This may be necessary for quantising general manifolds.

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## Appendix A: Some Results about the Universal Enveloping Algebra $\mathcal{U}$

This section is needed to establish the fact that  $P_n^m \in \mathcal{P}^n$ . Since the dimension of  $\mathcal{P}^n$  is  $2n + 1$  it is obvious that  $\{P_n^m, m = -n \dots n\}$  form an orthonormal basis for  $\mathcal{P}^n$ . (There may be an easier way without introducing this machinery). Some of these results are mentioned without proof in [13]

Let us define the algebra  $\mathcal{U}$  by

$$\mathcal{U} = \{\text{Free noncommuting algebra of polynomials in } x, y, z\} / \sim \quad (\text{A1})$$

where

$$[x, y] \sim i\kappa z, \quad [y, z] \sim i\kappa x, \quad [z, x] \sim i\kappa y, \quad (\text{A2})$$

There is a natural basis of this algebra given by the 3-vector  $S(a, b, c)$  with  $a, b, c \in \mathbb{Z}$  and  $a, b, c \geq 0$ . This represents the sum of all symmetric permutations of the word  $x^a y^b z^c$  each with coefficient 1. Thus for example

$$S(2, 1, 0) = x^2 y + x y x + y x^2$$

It is easy to show that  $S(a, b, c)$  has  $(a+b+c)!/(a!b!c!)$  terms. For consistency we define  $S(a, b, c) = 0$  if either  $a < 0$ ,  $b < 0$  or  $c < 0$ . These will be known as  $\mathcal{S}$ -expressions.

We can define the formal trace by

$$\begin{aligned} \text{Tr} : \mathcal{U} &\mapsto \mathcal{U} \\ \text{Tr} \left( \sum_{a_1 \dots a_p} f_{a_1 \dots a_p} x_{a_1} \cdots x_{a_p} \right) &= \sum_{b=1}^3 \sum_{a_1 \dots a_p} f_{b, b, a_3 \dots a_p} x_{a_3} \cdots x_{a_p} \end{aligned} \quad (\text{A3})$$

Thus if  $f \in \mathcal{U}$  and  $\text{Tr}(f) = 0$  then  $f \in \mathcal{P}$ . We have the following theorem for the manipulation of the  $\mathcal{S}$ -expressions.

**Theorem 15** *The formal trace of an  $\mathcal{S}$ -expression given by*

$$\text{tr} \left( S(a, b, c) \right) = S(a-2, b, c) + S(a, b-2, c) + S(a, b, c-2) \quad (\text{A4})$$

*The commutator of  $x$  and an  $\mathcal{S}$ -expression is given by*

$$e_x S(a, b, c) = [x, S(a, b, c)] = -\kappa(b+1)S(a, b+1, c-1) + \kappa(c+1)S(a, b-1, c-1) \quad (\text{A5})$$

*and cyclic permutation for  $e_y$ , and  $e_z$ . The relationship between these operations is given by*

$$\text{tr} \circ e_x = e_x \circ \text{tr} \quad (\text{A6})$$

*and similarly for  $e_y$ , and  $e_z$ . We can split an  $\mathcal{S}$ -expression to the  $m$  order to give*

$$S(a, b, c) = \sum_{d+e+f=m} S(a-d, b-e, c-f) S(d, e, f) \quad (\text{A7})$$

**Proof:**

We can think of the  $\mathcal{S}$ -expression  $S(a, b, c)$  as being a sum of terms. Let  $w$  be a permutation of  $x^d y^e z^f$ , with  $d \leq a, e \leq b, f \leq c$ . Take all the terms in  $S(a, b, c)$  which start with  $w$ . These terms must finish with each term in  $S(a-d, b-e, c-f)$ . This works with each permutation of  $x^d y^e z^f$  so  $S(a, b, c)$  must contain the term  $S(d, e, f)S(a-d, b-e, c-f)$ . Now if we let  $d, e, f$  run over all sets  $d+e+f=m$  and  $d, e, f \geq 0$  then this covers all possibilities, and no two are repeated. Hence (A7).

From (A7) putting  $m = 2$  we have

$$\begin{aligned} S(a, b, c) &= S(2, 0, 0)S(a-2, b, c) + S(0, 2, 0)S(a, b-2, c) + S(0, 0, 2)S(a, b, c-2) \\ &\quad + S(1, 1, 0)S(a-1, b-1, c) + S(1, 0, 1)S(a-1, b, c-1) + S(0, 1, 1)S(a, b-1, c-1) \end{aligned}$$

Now  $\text{tr}(x^2) = \text{tr}(y^2) = \text{tr}(z^2) = 1$  hence (A4). Proof of (A5) is by induction

on the order of the polynomial by the use of (A7) with  $m = 1$ . Proof is (A6) is by direct substitution. ■

**Corollary 16** *We are now in a position to prove that  $e_x : \mathcal{P}^n \mapsto \mathcal{P}^n$  and likewise for  $e_y, e_z, e_\pm, \Delta$ . Also  $P_n^m \in \mathcal{P}^n$ .*

**Proof of  $P_n^m \in \mathcal{P}^n$ :**

From the definition of  $P_n^n$

$$\begin{aligned} P_n^n &= \alpha_n J_+^n = \alpha_n (x + iy)^n \\ &= \alpha_n \sum_{r=0}^n i^r S(n-r, r, 0) \end{aligned}$$

also

$$\begin{aligned} \text{Tr}(P_n^n) &= \alpha_n \sum_{r=0}^n i^r (S(n-r-2, r, 0) + S(n-r, r-2, 0)) \\ &= \alpha_n \sum_{r=0}^{n-2} i^r S(n-r-2, r, 0) + \alpha_n \sum_{r=0}^{n-2} i^{r+2} S(n-r-2, r, 0) = 0 \end{aligned}$$

So  $P_n^n$  is a symmetric formally trace-free polynomial of order  $n$ . So  $P_n^n \in \mathcal{P}^n$ .

From (A5) we see that if  $f$  is an  $\mathcal{S}$ -expression of order  $n$  then so is  $e_-(f)$ . From (A6) we see that if  $f \in \mathcal{P}^n$  then  $e_-(f) \in \mathcal{P}^n$ . So  $P_n^m \in \mathcal{P}^n$ . ■

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